Five loop Konishi from AdS/CFT

Zoltán Bajnok[a,1], Árpád Hegedűs[b,2], Romuald A. Janik[c,3] and Tomasz Łukowski[c,4]

[a] Theoretical Physics Research Group, Hungarian Academy of Sciences, 1117 Budapest, Pázmány s. 1/A, Hungary
[b] Research Institute for Particle and Nuclear Physics, Hungarian Academy of Sciences, H-1525 Budapest 114, POB 49, Hungary
[c] Institute of Physics, Jagiellonian University, ul. Reymonta 4, 30-059 Kraków, Poland

Abstract

We derive the perturbative five loop anomalous dimension of the Konishi operator in N = 4 SYM theory from the integrable string sigma model by evaluating finite size effects using Lüscher formulas adapted to multimagnon states at weak coupling. In addition, we derive the five loop wrapping contribution for the $L = 2$ single impurity state in the β deformed theory, which may be within reach of a direct perturbative computation. The Konishi expression exhibits two new features - a modification of Asymptotic Bethe Ansatz quantization and sensitiveness to an infinite set of coefficients of the BES/BHL dressing phase. The result satisfies nontrivial self-consistency conditions - simple transcendentality structure and cancellation of μ-term poles. It may be a testing ground for the proposed AdS/CFT TBA systems.

[1] e-mail: `bajnok@elte.hu`
[2] e-mail: `hegedusa@rmki.kfki.hu`
[3] e-mail: `ufrjanik@if.uj.edu.pl`
[4] e-mail: `tomaszlukowski@gmail.com`

1 Introduction

The AdS/CFT correspondence [1] states the equivalence of type IIB superstrings on $AdS_5 \times S^5$ with the maximally supersymmetric $\mathcal{N} = 4$ gauge theory in four dimensions. Local operators in gauge theory correspond to states in superstring theory. In the planar limit, $N_c \to \infty$, multitrace operators factorize essentially into independent single trace factors which correspond, on the string side, to noninteracting strings. In this limit, one identifies the spectrum of a type IIB superstring in $AdS_5 \times S^5$ with the anomalous dimensions of single trace operators in the maximally supersymmetric $\mathcal{N} = 4$ four-dimensional $SU(N_c)$ gauge theory.

This identification should hold for any value of the 't Hooft coupling constant $\lambda = g_{YM}^2 N_c$. In the strong coupling limit, and for operators with sufficiently large charges, the corresponding strings may become classical/semiclassical and then the problem becomes tractable by conventional methods. In contrast, once the coupling is no longer large and/or we consider generic short operators the answer requires us to consider the worldsheet QFT of the superstring on the quantum level. In particular, if we would like to make contact with conventional perturbative gauge theory calculations and explicitly test the correspondence or follow some specific operator all the way from weak to strong coupling we would have to quantize exactly the worldsheet QFT of the superstring which is a highly nonlinear theory.

Fortunately, the worldsheet QFT of the superstring in the $AdS_5 \times S^5$ background is integrable, which was first shown on the classical level in [2], and assuming that integrability also holds on the quantum level (for which there are now many indications), one may use the theory of two-dimensional integrable quantum field theories to eventually quantize the theory in a quite explicit way for any value of λ. The AdS/CFT correspondence thus allows us to use tools and methods which exist only for *two-dimensional* quantum field theories to study the *four-dimensional* $\mathcal{N} = 4$ gauge theory.

The theory of two-dimensional integrable quantum field theories is well-developed by now. The general strategy to determine their finite volume spectrum goes as follows: First one considers the model in infinite volume. The Hilbert space of asymptotic states is built up from noninteracting multiparticle states which transform covariantly under the global symmetry algebra. Time evolution is formulated in terms of the scattering matrix that connects the initial and final multiparticle states [3]. Once integrability is assumed, there is no particle creation, and moreover any multiparticle scat-

tering process is shown to factorize into pairwise two particle scatterings. So the whole scattering information is contained in the $2 \rightarrow 2$ particle S-matrix. The requirements of unitarity, crossing symmetry, invariance under the global symmetry and the Yang-Baxter equation usually determines the scattering matrix uniquely up to CDD type ambiguities [4]. These are fixed through an analysis of the singularities of the scattering matrix all of which must have a physical origin. Thus the appearing poles have to correspond either to bound-states or to Coleman-Thun diagrams (anomalous thresholds). In this bootstrap solution we consider the bound-states and the original particles on equal footing and determine the scattering matrix of the bound-states from the scatterings on their individual constituents [5]. Then the singularity structure of the bound-state scattering matrix is analyzed and new bound-states are searched for. The so-called bootstrap program is completed if all singularities of all the scattering matrices are explained and then the theory is completely solved in infinite volume.

In the context of applications to the AdS/CFT correspondence, the infinite volume solution is not enough since one wants to describe closed strings and therefore consider the integrable quantum field theory on a cylinder of a given circumference related to a $U(1)_R$ charge of the given state. In contrast to conventional relativistic theories, *all* (integer [5]) sizes of the cylinder are in fact relevant for the complete spectrum of the theory.

The finite volume solution of the model can be achieved by systematically taking into account the finite size effects due to the scatterings of particles. The leading finite size effect of a multiparticle state comes from the quantization of momenta. It is described by Bethe-Yang equations and takes into account the scattering matrix in determining the allowed momenta [7]. It incorporates all polynomial corrections in the inverse of the volume. In addition, there are exponentially small (Lüscher) corrections as well and their leading contributions come from the polarization of the sea of virtual particles [8]. For small volumes, these effects become dominant and one needs to perform a resummation of the virtual corrections, which sometimes can be carried out in the form of nonlinear integral equations.

Indeed, the exact description of the finite volume ground state energy can be obtained from the Thermodynamic Bethe Ansatz (TBA) [9]. This is based on the fact that the contribution of the ground state energy domi-

[5] See also [6] where it was shown that the meromorphicity of Y-system leads to the quantization of the temperature of the mirror model.

nates the Euclidean partition function for large Euclidean time, and that the same partition function can be calculated by exchanging the role of space and time. One is left with the determination of the partition function at finite temperature, but in the large volume limit where finite size effects are under control. This method provides typically coupled integral equations for pseudo energies which determine the ground state energy exactly. In some circumstances a careful analytical continuation (in the volume say) can provide integral equations for excited states as well [10]. The procedure is a numerical one and the resulting equations are only conjectural which have to be further tested (but in all known cases they have passed all checks).

The analogous bootstrap solution of the $AdS_5 \times S^5$ worldsheet QFT of the light-cone quantized Green-Schwartz superstring is currently almost completed. Historically, the developments which led to it concentrated on the Bethe Ansatz part mostly on the gauge theory side [11, 12, 13, 14, 15] as well as for classical string solutions in $AdS_5 \times S^5$ [16, 17]. Later this was reformulated in the (spin-chain) S-matrix language in [18, 19], culminating in a proposal for the all loop Asymptotic Bethe Ansatz [19] the derivation of the (spin-chain) S-matrix from global symmetry properties [20]. In fact, this derivation could be transformed in a verbatim way into a starting point for the bootstrap solution of the worldsheet QFT. This was necessary since the worldsheet QFT perspective was crucial in order to tackle a new 'topological' class of Feynman graphs – 'wrapping diagrams' – which went beyond the Bethe Ansatz in the spin chain guise[6].

Let us therefore recall the basic steps of the bootstrap solution of the light-cone quantized Green-Schwartz string in $AdS_5 \times S^5$. It is classically integrable for any value of the light-cone momentum, P_+ identified with the $U(1)_R$ charge J, which serves as the volume of the two-dimensional theory. A notable new feature of this theory is that it is *not* relativistic invariant. In the decompactification limit $J \to \infty$ the massive excitation transform under the global symmetry algebra: the centrally extended $su_c(2|2)^2$. This symmetry [20], together with unitarity and crossing symmetry [22] completely fixes the scattering matrix[7] including the dressing factor [24, 25] modulo CDD ambiguities [26]. The analysis of the pole structure revealed an infinite tower of bound-states [27] whose scattering matrices have been calculated as well [28]. Double poles corresponding to anomalous thresholds were identified in [29]

[6]Note however the attempt to use the Hubbard model formalism for this purpose [21].

[7]Taking into account some important subtleties [23].

and the related Coleman-Thun diagrams were found. As the physical region of the AdS S-matrix is not known the complete analysis of its singularity structure is rigorously not completed yet.

In order to find the spectrum one has to pass to finite volume and consider the theory on a *cylinder*. The corrections to the Asymptotic Bethe Ansatz can be identified with wrapping corrections [30], and the leading Lüscher corrections for single [31] and multiparticle states [32] have been derived for this (nonrelativistic) theory. In this case the space-time interchange necessary for formulating TBA leads to quite a different theory [33] with a distinct set of bound-states. By now TBA equations are also developed by various groups [34, 35, 36], under various analyticity assumptions, in particular about the analytically continued dressing phase. There are conjectures also for excited states TBA equations [35] but they have not been tested yet beyond the leading perturbative order of wrapping corrections.

The formalism of the leading multiparticle Lüscher corrections have been quantitatively tested in the case of the 4-loop anomalous dimension of the Konishi operator where the leading order wrapping diagrams have been calculated perturbatively [37, 38]. The corresponding leading order Lüscher calculation found an excellent agreement [39]. Subsequently wrapping interactions computed from Lüscher corrections were found to be crucial for the agreement of some structural properties of twist two operators [39] with LO and NLO BFKL expectations [40].

The aim of the present paper is to elaborate the Lüscher correction to next to leading order and by this to calculate the anomalous dimension of the Konishi operator at five loops. Besides this explicit knowledge, which can serve as a testing ground for excited state TBA equations, we further test several issues of the formalism as well as a sizeable part of the BES/BHL dressing phase since the calculation requires the knowledge of the analytically continued dressing phase in the Lüscher kinematics. It relies on the conjectured finite size energy correction of multiparticle states, which at subleading order contains corrections originating from the modification of ABA. Although there is no perturbative gauge theory computation so far, the internal consistency of our calculation provides enough confirmation to believe in its correctness. In addition we consider the five loop subleading wrapping correction for a single impurity operator in the β deformed theory, which may be within reach of a direct perturbative verification.

The paper is organized as follows: In section 2 we summarize the main features of the 5-loop Konishi computation emphasizing the new phenom-

ena which appear w.r.t. the previous 4-loop case, and the possible internal consistency checks of the computation. In section 3 we rederive in a simple example the formalism of multiparticle Lüscher correction. We focus on a theory where the S-matrix does not depend on the difference of the rapidities and derive TBA equation for the ground state. Excited state energy levels are obtained by analytical continuation and, by analyzing their large volume asymptotics, multiparticle Lüscher corrections are extracted. As they contain the analytically continued scattering matrix, in Section 4 we determine the analytical continuation of the dressing phase by two different methods. Section 5 is devoted to the main computation of the anomalous dimension of the Konishi operator. It starts with the calculation of the ABA. Then we turn to the computation of the Lüscher correction. It has two sources: one comes from the modification of the ABA, the other corresponds to the virtual particles circulating around the cylinder. Both terms include an integration over the momentum of the mirror particles and a summation over their spectrum. Integration is carried out by residues, where, in comparison to the 4-loop case, we have to take into account an infinite tower of poles coming from the polygamma function part of the integrand. The resulting expression is composed of rational and polygamma functions which have to be summed over the bound-states. The technique developed for the summation is explained in Section 6. Finally we give our conclusions in Section 7. The paper is followed by two Appendices. In the first we calculate the 5-loop anomalous dimension of a single impurity operator, which acquires nontrivial wrapping corrections in the β deformed theory. In the second Appendix we explain how to sum up terms containing polygamma functions.

2 Main features of the 5-loop Konishi computation

The Konishi operator $\mathrm{tr}\,\Phi_i^2$ is the simplest operator not protected by supersymmetry which, thanks to its short size has proved to be a testing ground for the AdS/CFT correspondence. In most computations it is more convenient to use a different representative of the same supermultiplet which lies in the `sl(2)` sector

$$\mathrm{tr}\,(DZDZ) - \mathrm{tr}\,(ZD^2Z) \tag{1}$$

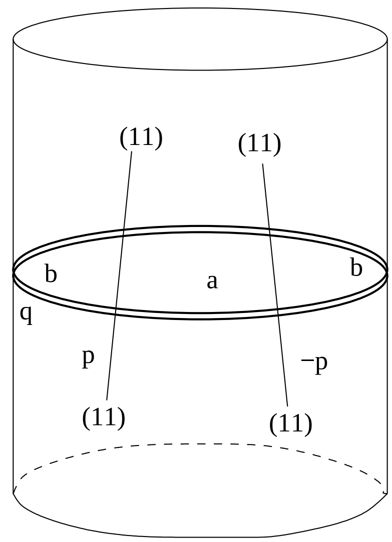

Figure 1: F-term of the Lüscher correction corresponds to virtual particles propagating around the worldsheet and scattering with the Konishi two-particle state, a and b denote internal states of the virtual particle while q is its 'mirror' momentum.

Its anomalous dimension following from ABA is given by[8]

$$\begin{aligned} E_{ABA} &= 4+12g^2-48g^4+336g^6-(2820+288\zeta(3))g^8 \\ &\quad +(26508+4320\zeta(3)+2880\zeta(5))g^{10}+\ldots \end{aligned} \tag{2}$$

However already at four loops there appears a contribution of wrapping graphs. The four loop wrapping contribution was computed directly in perturbative gauge theory [37] using supergraph techniques and reconfirmed together with the nonwrapping part using component Feynman graphs in [38]. In [39] the same result was computed from the string sigma model in $AdS_5 \times S^5$ and came from a Lüscher type F-term graph (see fig. 1) where a 'virtual' particle was circulating in a loop.

In the Lüscher corrections the virtual particle is strictly speaking on-shell, however its kinematics are from the space-time interchanged theory (so-called mirror theory). One can estimate its contribution to the magnitude of the

[8]See section 5.1 for a quick derivation.

expression for the wrapping correction which is [30]

$$e^{-2J \operatorname{arcsinh} \frac{\sqrt{1+q^2}}{4g}} \longrightarrow \frac{4^J g^{2J}}{(1+q^2)^J} \tag{3}$$

For the Konishi $J = 2$, so this gives a factor of g^4. Together with the so-called string frame phase factors [23] which promote the string length (light cone size of the worldsheet cylinder) to the 'spin chain length' these give another factor of g^4. Together the two factors give indeed g^8, making this contribution to appear at 4-loop. We expect therefore that the correction coming from two virtual particles should appear at least at order g^{12} (or if the string frame phase factors will also appear systematically this might be even as late as g^{16}). Therefore it is expected that the 5-loop wrapping part of the Konishi anomalous dimension will also appear from the leading Lüscher correction. What makes this computation interesting, and what is the main motivation for our study, is the fact that two new features which were absent in the 4-loop case make their appearance here.

Firstly, the dressing factor of the S-matrix between the mirror particle and the physical particles being the constituents of the Konishi state behaves like $\exp(ig^2\, phase)$, where the *phase* involves contribution from an *infinite* set of BES coefficients. This is in stark contrast to the behaviour of the dressing phase between physical particles where it behaves like $\exp(ig^6\, phase')$, and at higher orders the higher BES coefficients enter only one by one. We will explicitly compute the dressing phase between mirror and physical particles in section 4.

Secondly, the virtual particle will also modify ABA quantization condition. The reason that this effect appears only at 5-loop order is that the momenta of the constituent particles get shifted from the ABA value p_{ABA} by a term of order g^8:

$$p = p_{ABA} + g^8 \delta_w p_{ABA} \tag{4}$$

and hence their contribution to the energy coming from the dispersion relation would appear only starting from order g^{10}

$$E(p) = \sqrt{1 + 16g^2 \sin^2 \frac{p}{2}} = \sqrt{1 + 16g^2 \sin^2 \frac{p_{ABA}}{2}} + 4 \sin p_{ABA} \delta_w p_{ABA} g^{10} + \dots \tag{5}$$

The appearance of these two new effects is the main motivation for our calculation.

Of course the utility of this computation as a testing ground of the above two phenomena depends on the ability of having a cross-check. For the Konishi operator, a direct perturbative 5-loop computation seems to be beyond reach, however there are some very stringent internal crosschecks of the eventual final formula.

Firstly, from the structure of perturbative gauge theory integrals we expect the final answer to have a rather simple transcendentality structure – a linear combination of zeta functions (and possibly their products). Yet generically the subexpressions which appear when performing the computation from the string theory side are much more complicated like polygamma functions evaluated at irrational complex arguments. A nontrivial cross check will be the cancellation of these terms between the various parts of the computation, in particular between the contribution of the dressing phase, the modification of Asymptotic Bethe Ansatz and the direct F-term integral.

Secondly, purely from the integrable quantum field theory point of view we do not expect to have a contribution of so-called μ-terms due to the fact that at weak coupling bound-states are much heavier than fundamental particles. Typically μ-terms arise from certain 'dynamical' poles of the F-term integrand. A consequence of the vanishing of μ-term contribution would be a cancellation of the residues of those terms when summed over all bound-states. This again necessitates a subtle cancellation between the residues coming from the dressing phase, ABA modification and the F-term integrand. Therefore we may use these two consistency checks as a test both of our formalism and of a huge part of the BES dressing phase. Another motivation is the ongoing search and proposals for the nonlinear integral equations/functional equations which would exactly describe the spectrum for any size of the cylinder. The current computation could then be used as a test of these proposals especially as the modification of ABA quantization appears in a nontrivial way and the dressing phase expression in this regime is quite complicated.

We will also consider a single impurity state with momentum $p = \pi$ which can be considered as an analytical continuation of the twist-two operators considered at 4-loop in [32]. It should also coincide with a physical state in the β-deformed theory at $\beta = 1/2$. Such states have been considered perturbatively in [41] and it may be possible to have a direct 5-loop perturbative computation in this case. Here the modification of ABA quantization is absent, however there is a contribution from the infinite set of coefficients of the BES dressing phase. Therefore a direct perturbative computation would

be very interesting even for the single impurity case.

3 Multiparticle Lüscher formulas and ABA modification

In this section we will review the formalism of multiparticle Lüscher corrections introduced in [39]. In order to avoid ambiguities associated with the fact that the S-matrix for the $AdS_5 \times S^5$ superstring is a nontrivial function of both momenta and does not depend just on the difference of rapidities as is the case for relativistic integrable quantum field theories, we will consider the construction of the Thermodynamic Bethe Ansatz for a theory with diagonal scattering but with an S-matrix *without* the difference property. Then we will consider the construction of excited state TBA by analytical continuation along the lines of [10]. We will then extract the appropriate formulas for multiparticle Lüscher corrections by a large volume expansion recovering the expressions proposed in [39]. However we hope that the present derivation will make their origin clearer. We would like also to keep throughout the computation the conventions for the S-matrix predominantly used for the $AdS_5 \times S^5$ case which are different from the ones used usually for relativistic integrable QFT's.

3.1 TBA for diagonal scattering

The starting point for the derivation of the Thermodynamic Bethe Ansatz is the consideration of the mirror theory with space and time interchanged. We denote the momenta of the mirror theory by $\tilde{p}$ to make a clear distinction compared to the momenta of the original sigma model what we denote by p. In comparing with the results of [30] we note that $\tilde{p}$ was denoted there by p_{tba}. We consider a theory with one species of particles scattering with the S-matrix $S(\tilde{p}_1, \tilde{p}_2)$.

In this theory the ABA takes the form[9]

$$e^{i\tilde{p}_j R} = \prod_{k:k\neq j} S(\tilde{p}_j, \tilde{p}_k) \tag{6}$$

As is standard in TBA we are interested in the thermodynamic limit $R \to \infty$ of the free energy, where the physical size L is identified with the inverse

[9] We use conventions used in AdS/CFT literature.

temperature. We will parametrize the momenta by z (which may be identified with the coordinate on the rapidity torus). In the thermodynamic limit, the roots become very dense and can be substituted by continuous densities of roots $\rho(z)$ and densities of holes $\rho_h(z)$ (unoccupied roots cf. [9]).

Taking the logarithm of (6), and the derivative w.r.t. z we obtain a relation between $\rho(z)$ and $\rho_h(z)$:

$$2\pi(\rho(z)+\rho_h(z)) = R\tilde{p}'(z) - \frac{1}{i}\partial_z \log S(\tilde{p}(z),\cdot) * \rho \equiv R\tilde{p}'(z) - \phi * \rho \tag{7}$$

In order to get a second equation necessary for solving for both densities, we have to minimize the free energy which is given by

$$-RLf \equiv -LH + S = -L \int \tilde{E}(z)\rho(z)dz + S[\rho,\rho_h] \tag{8}$$

where $\tilde{E}$ is the mirror energy, while the entropy is

$$S[\rho,\rho_h] = \int dz \left\{(\rho+\rho_h)\log(\rho+\rho_h) - \rho\log\rho - \rho_h \log\rho_h\right\} \tag{9}$$

It is convenient to introduce the pseudoenergy $\varepsilon(z)$ which is related to the densities of roots through

$$\frac{\rho}{\rho+\rho_h} = \frac{e^{-\varepsilon}}{1+e^{-\varepsilon}} \tag{10}$$

and exremizing the free energy after taking into account the relation

$$\delta\rho + \delta\rho_h = -\frac{1}{2\pi}\phi * \delta\rho \tag{11}$$

coming from ABA, we obtain the final form of the TBA equation

$$\varepsilon(z) = L\tilde{E}(z) + \int \frac{dw}{2\pi}\,\phi(w,z)\log\left(1+e^{-\varepsilon(w)}\right) \tag{12}$$

We can now evaluate the thermal free energy of the mirror theory which gives the *physical* ground state energy of the original theory at zero temperature but at finite size.

$$E = Lf = -\int \frac{dz}{2\pi}\,\tilde{p}'(z)\log\left(1+e^{-\varepsilon(z)}\right) \tag{13}$$

3.2 TBA for excited states by analytical continuation

Initially, the Thermodynamic Bethe Ansatz could give information only on the exact ground state energy, and it was not known how to extract similar information for excited states in finite volume. In [10], a version of the Thermodynamic Bethe Ansatz for excited states was constructed by making an analytical continuation of the ground state equations. The idea was that in deforming contours of integration in the TBA equations, zeroes of $1+e^{-\varepsilon(z)}$ would generate poles in the integrand which could then be rewritten as additional source terms of the equations. Here we will perform an analogous heuristic procedure, fixing the sign of the pole contribution (which depends on the orientation of the contour w.r.t. the singularities) in order to reproduce ABA results at large volume. Once this is fixed, we will be able to unambiguously define leading corrections. Note that we assume here that there are no μ-terms so the calculation is analogous to the Sinh-Gordon model in the relativistic case and so we will assume that each physical particle is represented just by a single pole.

We will need the relation between the mirror energies and momenta $(\tilde{E},\tilde{p})$ and the physical ones (E,p). These are defined by

$$\tilde{E} = ip \qquad\qquad \tilde{p} = iE \tag{14}$$

which is fixed by taking the mirror integral to be for $z = x - \omega_2/2$ with x real, and consequently identifying z^-/z^+ with $e^{-\tilde{E}}$. An opposite choice is also possible.

Let us first consider the energy formula (13) integrated by parts:

$$E = \int \frac{dz}{2\pi}\, \tilde{p}(z)\partial_z \log\left(1+e^{-\varepsilon(z)}\right) \tag{15}$$

Suppose that $1+e^{-\varepsilon(z_{1,2})}=0$. Then the orientation has to be such that the $\partial_z \log\left(1+e^{-\varepsilon(z)}\right)$ contributes -1 by residues. We thus obtain

$$E = E(z_1)+E(z_2) - \int \frac{dz}{2\pi}\, \tilde{p}'(z)\log\left(1+e^{-\varepsilon(z)}\right) \tag{16}$$

By the same mechanism an analogous contribution will arise from the TBA equation (12), with the sign already fixed by the above considerations. We get

$$\varepsilon(z) = L\tilde{E} + \log S(z_1,z) + \log S(z_2,z) + \int \frac{dw}{2\pi i}\left(\partial_w \log S(w,z)\right)\log\left(1+e^{-\varepsilon(w)}\right) \tag{17}$$

The multiparticle Lüscher formulas will follow, for this theory, by performing large volume (large L) expansion.

3.3 Multiparticle Lüscher formulas

Firstly in order to compute the energy keeping the first exponential corrections, we may neglect the integral term of (17) when inserting $\varepsilon(z)$ into (16). We thus get

$$\begin{aligned} E &= E(z_1) + E(z_2) - \int \frac{dz}{2\pi}\, \tilde{p}' e^{-L\tilde{E}} \frac{1}{S(z_1, z)S(z_2, z)} \\ &= E(z_1) + E(z_2) - \int \frac{d\tilde{p}}{2\pi}\, e^{-L\tilde{E}} S(z, z_1)S(z, z_2) \end{aligned} \tag{18}$$

We recognize at once the F-term integral (with $q \equiv \tilde{p}$).

In order to complete the formula we need to self consistently fix the positions of the poles z_1 and z_2. We will do it in two steps. First we neglect the integral term of (17) and impose for the rest $\varepsilon(z_i) = i\pi + (2\pi n)i$. We get

$$i\pi = \varepsilon(z_1) = iLp_1 + i\pi + \log S(z_2, z_1) \tag{19}$$

where we supposed that $S(z, z) = -1$. This gives at once ABA equation with our conventions

$$e^{iLp_1} = S(p_1, p_2) \tag{20}$$

However it turns out that we have to be more precise in the determination of the position of the roots and we have to include the integral term. In order to define the quantization conditions we thus have to use

$$\begin{aligned} \varepsilon(z) &= iLp(z) + \log S(z_1, z) + \log S(z_2, z) + \\ &\quad + \int \frac{dw}{2\pi i}\, \frac{\partial_w S(w, z)}{S(w, z)} e^{-L\tilde{E}(w)} S(w, z_1)S(w, z_2) \end{aligned} \tag{21}$$

The quantization conditions $\varepsilon(z_i) = i\pi$ takes the form

$$0 = \underbrace{\log\{e^{iLp_1} S(z_2, z_1)\}}_{BY_1} + \underbrace{\int \frac{dw}{2\pi i}\, (\partial_w S(w, z_1))S(w, z_2)e^{-L\tilde{E}(w)}}_{\Phi_1} \tag{22}$$

$$0 = \underbrace{\log\{e^{iLp_2} S(z_1, z_2)\}}_{BY_2} + \underbrace{\int \frac{dw}{2\pi i}\, S(w, z_1)(\partial_w S(w, z_2))e^{-L\tilde{E}(w)}}_{\Phi_2} \tag{23}$$

Since the integrals are exponentially small we may solve these equations in terms of corrections to ABA giving

$$\begin{aligned} \frac{\partial BY_1}{\partial p_1}\delta p_1 + \frac{\partial BY_1}{\partial p_2}\delta p_2 + \Phi_1 &= 0 \qquad (24) \\ \frac{\partial BY_2}{\partial p_1}\delta p_1 + \frac{\partial BY_2}{\partial p_2}\delta p_2 + \Phi_2 &= 0 \qquad (25) \end{aligned}$$

The final formula for the energy thus takes the form

$$E = E(p_1)+E(p_2)+E'(p_1)\delta p_1+E'(p_2)\delta p_2-\int \frac{dq}{2\pi}\, e^{-L\tilde{E}} S(z,z_1)S(z,z_2) \qquad (26)$$

For the case of the $AdS_5 \times S^5$ string sigma model we just have to replace the product of the S-matrices by an appropriate supertrace. Thus the integrand becomes essentially the transfer matrix. However the part coming from the modification of ABA quantization does not have *a-priori* such a direct relation to the transfer matrix. Hence in the following we will evaluate it directly from its definition using the S-matrices and their derivatives.

4 The dressing phase in the Lüscher kinematics

As we explained in section 2 one of the main sources of corrections which appears at five loops is the contribution of the dressing phase. In this section we present two calculations for its leading order part in the Lüscher kinematics. By this we mean a kinematics which is relevant in calculating the finite size correction, that is when the first argument is in the mirror region $|x_1| < 1$ while the second is in the physical one $|x_2| > 1$.

The dressing phase when both arguments are in the physical region, $(|x^\pm_{1,2}| > 1)$ can be written in the following form

$$\theta(x_1,x_2) = \chi(x_1^+,x_2^+) - \chi(x_1^+,x_2^-) - \chi(x_1^-,x_2^+) + \chi(x_1^-,x_2^-) \qquad (27)$$

where

$$\chi(x_1,x_2) = -\sum_{r=2}^{\infty}\sum_{s>r} \frac{c_{r,s}(g)}{(r-1)(s-1)} \left[\frac{1}{x_1^{r-1}x_2^{s-1}} - \frac{1}{x_1^{s-1}x_2^{r-1}}\right] \qquad (28)$$

The coefficients $c_{r,s}(g)$ have a convergent weak coupling expansion [25]

$$c_{r,s}(g) = 2\cos\left(\frac{\pi}{2}(s-r-1)\right)(r-1)(s-1)\int_0^\infty dt \frac{J_{r-1}(2gt)J_{s-1}(2gt)}{t(e^t-1)} \quad (29)$$

As was shown in [29] the dressing phase has also an alternative double integral representation, which is suitable for analytical continuations:

$$\chi(x_1,x_2) = i\oint_{C_1}\frac{dw_1}{2\pi i}\frac{1}{w_1-x_1}I_2(w_1,x_2) \quad (30)$$

where

$$I_2(w_1,x_2) = \oint_{C_1}\frac{dw_2}{2\pi i}\frac{1}{w_2-x_2}\log\frac{\Gamma(1+ig(w_1+w_1^{-1}-w_2-w_2^{-1}))}{\Gamma(1-ig(w_1+w_1^{-1}-w_2-w_2^{-1}))} \quad (31)$$

and the integrations go over the unit circles. In the next subsections we focus on the weakly coupled regime, review the dressing phase for physical particles in this regime and calculate the analytical continuation for both representations in the Lüscher kinematics.

4.1 BES dressing phase for physical particles

In the $g \to 0$ limit we can expand the Bessel function as

$$J_n(2gt) = \frac{g^n t^n}{n!}(1 - \frac{t^2 g^2}{n+1} + \dots) \quad (32)$$

and perform the integration in (29). This provides the leading order behaviour of $c_{r,s}(g)$:

$$\begin{aligned} c_{r,s}(g) &= \frac{2\cos(\frac{\pi}{2}(s-r-1))}{(r-2)!(s-2)!}g^{r+s-2}\big[(r+s-3)!\zeta(r+s-2) + \\ &\quad -\frac{g^2}{rs}(r+s)(r+s-1)!\zeta(r+s) + \dots\big] \end{aligned} \quad (33)$$

In calculating the Lüscher correction we have to analyze the dressing phase (27) for the case when the parameters x_1, x_2 in the $g \to 0$ limit behave as

$$x_1^+ = \frac{q+iQ}{2g} + O(g) \quad ; \quad x_1^- = \frac{2g}{(q-iQ)} + O(g^3) \quad ; \quad x_2^\pm = \frac{2u \pm i}{2g} + O(g^3) \quad (34)$$

Let us start with the case when both parameters are in the physical kinematics $x_{1,2} = a_{1,2}g^{-1} + O(g)$. Then the leading order behaviour comes from the term $c_{2,3}(g)$ and is given by

$$\chi(x_1, x_2) = -(2g^6\zeta(3) + 20g^8\zeta(5))\frac{a_1 - a_2}{a_1^2 a_2^2} + O(g^{10}) \tag{35}$$

The next coefficients $c_{2,5}(g)$ and $c_{3,4}(g)$ only enter at g^{10} order. For physical particles, the dressing phase starts at order g^6 and this is what we have to use in the calculation of the ABA.

Let us now describe the regime where one of the particles is in the Lüscher ('mirror') kinematic which is relevant for the five loop wrapping computation.

4.2 BES dressing phase in the Lüscher kinematics

Suppose now that the first argument is in the mirror kinematics $x_1 = a_1^{-1}g + O(g^3)$ while we keep the second in the physical one $x_2 = a_2 g^{-1} + O(g)$. Since $c_{r,s}(g)$ scales as g^{r+s-2} while $x_1^{1-r}x_2^{1-s}$ as g^{s-r} the leading order contribution comes from the $r = 2$ part of the first sum:

$$\chi(x_1, x_2) = \sum_{s>2} \frac{c_{2,s}(g)}{(s-1)} \frac{1}{x_1^{s-1} x_2} + O(g^4) \tag{36}$$

Clearly this is in stark contrast to the case of the ABA as in this Lüscher kinematics infinite number of coefficients contribute. Using the leading order term of the explicit weak coupling expansion of $c_{2,s}$ determined in (33) we obtain

$$\begin{aligned}
\chi(x_1, x_2) &= \frac{g^2}{a_2} \sum_{s>2} 2\cos(\frac{\pi}{2}(s-3)) a_1^{s-1} \zeta(s) + O(g^4) \\
&= \frac{g^2}{a_2}(S_1(-ia_1) + S_1(ia_1)) + O(g^4)
\end{aligned}$$

where $S_1(n) = \sum_{k=1}^n \frac{1}{k}$ is the harmonic number which has an analytical continuation in terms of the digamma function $\psi(x) = \frac{d \log \Gamma(x)}{dx}$ as

$$\begin{aligned}
\chi(x_1, x_2) &= \frac{g^2}{a_2}(2\gamma_E + \psi(1 - ia_1) + \psi(1 + ia_1)) + O(g^4) \\
&= \frac{g^2}{a_2}(2\gamma_E + \psi(-ia_1) + \psi(ia_1)) + O(g^4)
\end{aligned} \tag{37}$$

where γ_E is the Euler constant. Thus for sufficiently small a_1 the original BES dressing phase provided a convergent expansion for the leading order $g \to 0$ behaviour in the Lüscher kinematics. The resulting expression then can be analytically continued for any a_1. Let us check that this is the right analytical continuation by calculating the weak coupling expansion of the analytically continued dressing phase from the DHM integral representation.

4.3 BES dressing phase in the Lüscher kinematics from the DHM integral formula

The proper analytical continuation of the dressing phase (30) into the Lüscher kinematics, compatible with crossing symmetry, was determined in [42]:

$$\chi(x_1, x_2) = i \oint_{C_1} \frac{dw_1}{2\pi i} \frac{1}{w_1 - x_1} I_2(w_1, x_2) - i I_2(x_1, x_2) \tag{38}$$

As in the previous subsection we are interested in the weak coupling expansion of χ in the Lüscher kinematics so we take $x_1 = a_1^{-1} g + O(g^3)$ and $x_2 = a_2 g^{-1} + O(g)$. Observe that since $|x_2| > 1$ the contour never encircles x_2. Let us focus on the second term. We change the integration variable to $u_2 = g w_2$ and write

$$I_2(x_1, x_2) = \oint_{C_g} \frac{du_2}{2\pi i} \frac{1}{u_2 - a_2} \log \frac{\Gamma(1 + ig(x_1 + x_1^{-1}) - iu_2 - ig^2 u_2^{-1}))}{\Gamma(1 - ig(x_1 + x_1^{-1}) + iu_2 + ig^2 u_2^{-1}))} \tag{39}$$

Since the integration for u_2 goes over a shrinking circle of radius $g \to 0$ we pick up the contributions of the poles at $u_2 = 0$ only (not at a_2). From the expansion of the Γ function in $g^2 u_2^{-1}$ one can observe that higher order poles contribute at higher orders in g^2. Thus when we focus on the leading order behaviour we are allowed to keep the first order pole:

$$I_2(x_1, x_2) = i \left(\frac{g^2}{a_2} \right) \left[\psi(1 + ig(x_1 + x_1^{-1})) + \psi(1 - ig(x_1 + x_1^{-1})) \right] + O(g^4) \tag{40}$$

Taking into account that $x_1 = a_1^{-1} g + O(g^3)$, the leading order part turns out to be

$$I_2(x_1, x_2) = i \left(\frac{g^2}{a_2} \right) \left[\psi(1 + ia_1) + \psi(1 - ia_1) \right] + O(g^4) \tag{41}$$

In addition to this term we also have a term of the form

$$I_1(x_1, x_2) = i \oint_{C_1} \frac{dw_1}{2\pi i} \frac{1}{w_1 - x_1} I_2(w_1, x_2) \tag{42}$$

Since $x_1 = a_1^{-1} g + O(g^3)$, it lies inside the integration contour. We need a convergent small coupling expansion. For this we change the integration variable to $u_1 = w_1 g^{-1}$ and close the contour around ∞. The leading oder result comes only from the pole at ∞ and reads as:

$$I_1(x_1, x_2) = \left(\frac{g^2}{a_2}\right) 2\gamma_E + O(g^4) \tag{43}$$

In order to determine the leading order behaviour of χ we have to combine I_1 and I_2. The result provides the leading order asymptotics of the dressing phase in the Lüscher kinematics

$$\chi(x_1, x_2) = \left(\frac{g^2}{a_2}\right) [2\gamma_E + \psi(1 + ia_1) + \psi(1 - ia_1)] + O(g^4) \tag{44}$$

which is consistent with what we obtained from the BES dressing phase. This is the main result of this section. We will need to take into account this expression when we calculate the five loop anomalous dimension of the Konishi operator in the next section.

5 The Konishi computation

In this section, which is the main part of the paper we describe and evaluate the various ingredients which together contribute to the 5-loop wrapping correction. For completeness we first describe the evaluation of the Asymptotic Bethe Ansatz (ABA) contribution to the anomalous dimension and then proceed to compute the various parts of the wrapping correction, summarizing the previous 4-loop result in the process.

5.1 Asymptotic Bethe Ansatz for the Konishi operator

In this subsection we calculate the scaling dimension of the Konishi operator from the ABA. In principle the complete scaling dimension has the form

$$\Delta(g) = \Delta_{ABA}(g) + \Delta_w(g) \tag{45}$$

where the first term contains the result of the ABA, while the second the wrapping contributions. In doing the calculation we choose the representative for the Konishi operator in the sl_2 sector. ABA amounts to calculate in this case the energy of the two particle state in volume $L = 2$ with momentum p and $-p$ which satisfy the equation

$$e^{ipL} = S(p, -p) = S^{sl(2)}(p, -p)e^{2i\theta(x(p),x(-p))} \tag{46}$$

where $S^{sl(2)}(p, -p)$ is the scattering matrix in the sl_2 sector which reads as

$$S^{sl(2)}(x^{\pm}(p), x^{\pm}(-p)) = \frac{x^-(p) - x^+(-p)}{x^+(p) - x^-(-p)} \frac{1 - \frac{1}{x^+(p)x^-(-p)}}{1 - \frac{1}{x^-(p)x^+(-p)}} \tag{47}$$

and $x^{\pm}(p) = \frac{1}{4g}(\cot(\frac{p}{2}) \pm i)(1 + E(p))$. We can solve this equation perturbatively. We take the Ansatz $p = \sum_i p^{(i)} g^{2i}$ and systematically expand both the scalar and the dressing part in increasing orders of g. The first two orders can be determined without the dressing phase. These solutions then can be put into formula (35) to determine the leading contributions of the dressing phase. Finally we find the solution for p at the required order

$$\begin{aligned} p \;&=\; \frac{2\pi}{3} - \sqrt{3}g^2 + \frac{9\sqrt{3}}{2}g^4 - 24\sqrt{3}(1 + \zeta(3))g^6 \\ &\quad + \frac{\sqrt{3}}{4}(671 + 960(\zeta(3) + \zeta(5)))g^8 + O(g^{10}) \end{aligned} \tag{48}$$

This momentum then determines the ABA part of the scaling dimension as

$$\begin{aligned} \Delta_{ABA}(g) \;&=\; 2E(p) = 4 + 12g^2 - 48g^4 + 336g^6 - (2820 + 288\zeta(3))g^8 \\ &\quad + (26508 + 4320\zeta(3) + 2880\zeta(5))g^{10} + O(g^{12}) \end{aligned}$$

The wrapping part of the anomalous dimension starts as

$$\Delta_w(g) = \Delta_w^{(8)} g^8 + \Delta_w^{(10)} g^{10} + O(g^{12}) \tag{49}$$

In [39] the 4-loop part has been evaluated to give

$$\Delta_w^{(8)} = 324 + 864\zeta(3) - 1440\zeta(5) \tag{50}$$

and the aim of the rest of this paper is to calculate the wrapping part at 5-loop order $\Delta_w^{(10)}$. Our final result is

$$\Delta_w^{(10)} = -11340 + 2592\zeta(3) - 5184\zeta(3)^2 - 11520\zeta(5) + 30240\zeta(7) \tag{51}$$

5.2 The general structure of the wrapping correction

There are two sources of the Lüscher correction for the Konishi operator which have to be taken into account when we calculate the 5-loop anomalous dimension (recall formula (26)).

Modification of Asymptotic Bethe Ansatz quantization

First of all the leading contribution which comes from the modification of ABA is of order g^{10}. This has the form

$$\Delta_w^{ABA} = E'(p_1)\delta p_1 + E'(p_2)\delta p_2 \tag{52}$$

where $p_1 = -p_2 \equiv p$ is the solution (48) of the ABA, while δp_i are the shifts due to the virtual corrections. For the 5-loop result it will be enough to take $p = 2\pi/3$ in this formula.

The shifts δp_i for the Konishi operator can be found from relations (24) and (25) which in our case take the form

$$\begin{aligned} \frac{5i}{2}\delta p_1 - \frac{i}{2}\delta p_2 + \Phi_1 &= 0 \\ -\frac{i}{2}\delta p_1 + \frac{5i}{2}\delta p_2 + \Phi_2 &= 0 \end{aligned}$$

where $\Phi = \Phi_1 = -\Phi_2$ and Φ for the Konishi operator is as follows

$$\begin{aligned} i\Phi &= \sum_Q \int_{-\infty}^{\infty} \frac{dq}{2\pi} \left(\frac{z^-}{z^+}\right)^L \sum_b (-1)^{F_b} \left[(\partial_q S_{Q-1}(q,u_i)) S_{Q-1}(q,u_{ii})\right]_{b(11)}^{b(11)} \\ &\equiv \sum_{Q=1}^{\infty} \int_{-\infty}^{\infty} \frac{dq}{2\pi}\, \Phi_Q(q,u) \end{aligned} \tag{53}$$

Solving the equations for δp_1 and δp_2 we found that

$$\delta p_1 = -\delta p_2 = \frac{i}{3}\Phi$$

This means that the momenta of the Konishi constituents are shifted in opposite directions by the same factor. It leads to the vanishing of the total momentum after ABA modification as it should be.

When plugged into the dispersion relation it gives us the energy shift (52) due to the Asymptotic Bethe Ansatz modification as

$$\Delta_w^{ABA} = \frac{4}{\sqrt{3}}\Phi \tag{54}$$

We will evaluate Φ explicitly later in the section.

The F-term integral

The second contribution to the 5-loop result comes from the expansion of formula [39]

$$\Delta_w^F = -\sum_{Q=1}^{\infty}\int_{-\infty}^{\infty}\frac{dq}{2\pi}\left(\frac{z^-}{z^+}\right)^L\sum_b(-1)^{F_b}\left[S_{Q-1}(q,u_i)S_{Q-1}(q,u_{ii})\right]_{b(11)}^{b(11)} \tag{55}$$

up to subleading terms. We parametrize the virtual particle using the mirror momentum q as in [39] (see also (68)), while for the physical particles we use rapidities as in [32]. Hence $u_i = -u_{ii} \equiv u$ are the rapidities of the two particles forming the Konishi state. They are related to the momentum p as

$$u(p) = \frac{1}{2}\cot(\frac{p}{2})\sqrt{1+16g^2\sin^2(\frac{p}{2})} \tag{56}$$

The rapidity variable for the Konishi state is therefore given as

$$u = u_0 + u_2 g^2 + \ldots = \frac{1}{2\sqrt{3}} + \frac{4}{\sqrt{3}}g^2 + \ldots \tag{57}$$

To simplify our further considerations we rewrite (55) in the form

$$\begin{aligned}\Delta_w^F &= -\sum_{Q=1}^{\infty}\int_{-\infty}^{\infty}\frac{dq}{2\pi}\Big(\frac{z^-}{z^+}\Big)^L S_0(q,Q,u)S_\sigma(q,Q,u)S_{\boxplus}(q,Q,u)^2 \\ &= -\sum_{Q=1}^{\infty}\int_{-\infty}^{\infty}\frac{dq}{2\pi}Y_Q(q,u)\end{aligned} \tag{58}$$

where we split the contribution coming from the S-matrix into three parts: the scalar, the dressing and the matrix part. Each of those parts involves a product over contributions of the two particles forming the Konishi state.

It is convenient to perform the weak coupling expansion of this formula in two steps. First we keep u as a variable of order 1 and expand all other sources of g dependence. Then we take into account the fact that u itself is g dependent and compute the g dependent terms due to (57).

All ingredients in formula (58) can be expanded into a series in g^2, keeping u fixed, as

$$\begin{aligned}
\left(\frac{z^-}{z^+}\right)^2 &= g^4\Upsilon^{(4)}(q,Q) + g^6\Upsilon^{(6)}(q,Q) + \ldots \\
S_0(q,Q,u) &= S_0^{(0)}(q,Q,u) + g^2 S_0^{(2)}(q,Q,u) + \ldots \\
S_\sigma(q,Q,u) &= 1 + g^2 S_\sigma^{(2)}(q,Q,u) + \ldots \\
S_\boxplus(q,Q,u) &= g^2 S_\boxplus^{(2)}(q,Q,u)^2 + g^4 S_\boxplus^{(4)}(q,Q,u) + \ldots
\end{aligned}$$

Now we will factor out the leading piece

$$Y_Q^{(8)}(q,u)g^8 = S_0^{(0)}(q,Q,u)S_\boxplus^{(2)}(q,Q,u)^2\Upsilon^{(4)}(q,Q,u)g^8 \tag{59}$$

and rewrite the first subleading term as a sum of contributions coming from the matrix part, scalar part, the exponential term and the dressing factor:

$$\begin{aligned}
Y_Q^{(10)}(q,u) &= Y_Q^{(8)}(q,u)\Big[2\frac{S_\boxplus^{(4)}(q,Q,u)}{S_\boxplus^{(2)}(q,Q,u)} + \frac{S_0^{(2)}(q,Q,u)}{S_0^{(0)}(q,Q,u)} \\
&\qquad + \frac{\Upsilon^{(6)}(q,Q)}{\Upsilon^{(4)}(q,Q)} + S_\sigma^{(2)}(q,Q,u)\Big]
\end{aligned}$$

In the following we will calculate all these contributions one by one.

Since $Y_Q^{(10)}(q,u)$ is already at order g^{10}, we may safely set here $u = \frac{1}{2\sqrt{3}}$.

$$Y_Q^{(10)}(q,u)g^{10} = Y_Q^{(10,0)}(q)g^{10} + \mathcal{O}\left(g^{12}\right) \tag{60}$$

where $Y_Q^{(10,0)}(q) = Y_Q^{(10)}(q,u_0)$.

As mentioned earlier we also have to take into account that the rapidity u is g dependent by itself. Clearly the only part that will contribute at order g^{10} is the expansion of (59)

$$Y_Q^{(8)}(q,u)g^8 = Y_Q^{(8,0)}(q)g^8 + Y_Q^{(8,2)}(q)g^{10} + \ldots \tag{61}$$

with $Y_Q^{(8,0)}(q) = Y_Q^{(8)}(q, u_0)$. To summarize, the wrapping correction at 4- and 5-loop respectively take the form

$$\Delta_w^{(8)} = -\sum_{Q=1}^{\infty} \int_{-\infty}^{\infty} \frac{dq}{2\pi} Y_Q^{(8,0)}(q) \tag{62}$$

$$\Delta_w^{(10)} = -\sum_{Q=1}^{\infty} \int_{-\infty}^{\infty} \frac{dq}{2\pi} \left(\frac{4}{\sqrt{3}} \Phi_Q(q) + Y_Q^{(10,0)}(q) + Y_Q^{(8,2)}(q) \right) \tag{63}$$

The leading contribution and 4-loop wrapping

The leading expansion of integrand $Y_Q(q, u)$ is known to be proportional to g^8 and was found in [39] as

$$\begin{aligned} Y_Q^{(8)}(q, u) g^8 &= S_0^{(0)}(q, Q, u) S_{\boxplus}^{(2)}(q, Q, u)^2 \Upsilon^{(4)}(q, Q, u) g^8 \\ = \frac{16384 g^8 Q^2 (-1 + q^2 + Q^2 - 4u^2)^2}{(q^2 + Q^2)^4 ((q + i(Q+1))^2 - 4u^2)((q + i(Q-1))^2 - 4u^2)} &\times \\ \times \frac{1}{((q - i(Q-1))^2 - 4u^2)((q - i(Q+1))^2 - 4u^2)} \end{aligned}$$

When expanded further using the fact that the rapidity variable u is g^2-dependent (57) it can be rewritten as

$$Y_Q^{(8)}(q, u) = Y_Q^{(8,0)}(q) + g^2 Y_Q^{(8,2)}(q) \tag{64}$$

where ${Y_Q}^{(8,0)}(q)$ is the only contribution which is relevant for the 4-loop calculations.

$$\Delta_w^{(8)} = -\sum_{Q=1}^{\infty} \int_{-\infty}^{\infty} {Y_Q}^{(8,0)}(q) \tag{65}$$

The integrand is a rational function and hence the integral can be carried out by residues. The poles are of two kinds. Firstly there is a fourth order pole at $q = iQ$ which we call ‘kinematical’ pole since it comes just from the exponential terms. The remaining poles come from the S-matrix and are ‘dynamical’ i.e. s and t channel poles. They would be associated to possible μ-terms. Since, as argued in [39], μ-terms should not be present at weak coupling, we expect that the residues of the dynamical poles should sum up to zero after summation over all Q. We can indeed verify that this is the case.

Hence the 4-loop wrapping correction to the Konishi anomalous dimensions can be obtained just by summing the residue of the kinematical pole only:

$$\Delta_w^{(8)} = -2\pi i \sum_{Q=1}^{\infty} \operatorname*{res}_{q=iQ} Y_Q^{(8,0)}(q) = 324 + 864\zeta(3) - 1440\zeta(5) \tag{66}$$

We expect that the same cancellation of 'dynamical' poles will also hold at 5-loop. Indeed this will be a nontrivial internal consistency check of our calculation.

5.3 The 5-loop integrand

The full integrand which will be relevant for 5-loop calculations consists of the subleading contribution of $Y_Q^{(8)}(q,u)$ (namely $Y_Q^{(8,2)}(q)$), the leading contribution of $Y_Q^{(10)}(q,u)$ (namely $Y_Q^{(10,0)}(q)$ which comes from the matrix, scalar, exponential part and from the dressing phase) and the leading contribution $\Phi_Q(q)$ coming from the ABA modification. We will now evaluate one by one the individual contributions.

Matrix part

The subleading matrix part of integrand $S_{\boxplus}^{(4)}(q,Q,u)$ for the Konishi operator can be evaluated using the formulas (78)-(82) from [39]. When normalized with respect to leading term $S_{\boxplus}^{(2)}(q,Q,u)$ it stands as

$$\begin{aligned}\frac{S_{\boxplus}^{(4)}(q,Q,u)}{S_{\boxplus}^{(2)}(q,Q,u)} &= \frac{16q(1-iq-Q-4u^2)}{(q+iQ)((q-i(Q-1))^2-4u^2)(1+4u^2)} - \frac{16}{(1+4u^2)^2} \\ &+ \frac{16(q^2+Q^2-1)}{(q^2+Q^2)(q^2+Q^2-1-4u^2))} + \frac{4(-5+5q^2+4Q+5Q^2-4u^2)}{(q^2+Q^2)(1+4u^2)}\end{aligned}$$

Scalar part

The scalar part of the integrand can be evaluated using the S-matrices of the scattering of the `sl(2)` bound-state constituents (z_1^-, z_1^+), ..., (z_Q^-, z_Q^+) with the fundamental magnon (x^-, x^+):

$$S_0(z^\pm, x^\pm) = \prod_{i=1}^{Q} S^{sl(2)}(z_i^\pm, x^\pm) \tag{67}$$

with

$$S^{sl(2)}(z^{\pm},x^{\pm})=\frac{z^{-}-x^{+}}{z^{+}-x^{-}}\frac{1-\frac{1}{z^{+}x^{-}}}{1-\frac{1}{z^{-}x^{+}}}$$

In order to calculate (67) we use for z_1^- the value z^- and for z_Q^+ the value z^+ where

$$z^{\pm}=\frac{Q}{4g}\left(-\sqrt{1+\frac{16g^2}{Q^2+q^2}}\mp 1\right)\left(-\frac{q}{Q}-i\right) \tag{68}$$

whereas z_k^+ we take from

$$z_k^+=\frac{1}{2}\left(z_k^-+\frac{1}{z_k^-}+\frac{i}{g}+\sqrt{\left(z_k^-+\frac{1}{z_k^-}+\frac{i}{g}\right)^2-4}\right)$$

The intermediate z_k^- are determined from the pole condition $z_k^-=z_{k-1}^+$. In principle we could choose different signs in front of the square roots. This amounts to choosing a different representative for the constituents of the Q bound-state. It was shown however in [42] that all of the different constituents lead to the same S-matrix after analytical continuation. Our choice is technically the simplest one and leads to the expansion of the bound-state parameters up to the second order as

$$z_k^+=\frac{2ik+q-iQ}{2g}+\frac{2g}{q-iQ}-\frac{2g}{2ik+q-iQ}+\frac{2g}{q+iQ} \tag{69}$$

The result for the leading order scalar part of the integrand is then

$$\begin{aligned}S_0^{(0)}(q,Q,u) &= \frac{((q-i(Q-1))^2-4u^2)(1+4u^2)^2}{((q-i(Q+1))^2-4u^2)((q+i(Q+1))^2-4u^2)}\times\\ &\qquad\times\frac{1}{((q+i(Q-1))^2-4u^2)}\end{aligned}$$

while the subleading one can be split into two parts: a rational part

$$\begin{aligned}\frac{S_{0rat}^{(2)}(q,Q,u)}{S_0^{(0)}(q,Q,u)} &= -\frac{32}{(iq+Q)(1+4u^2)}+\frac{32}{(1+4u^2)^2}+\\ &+\frac{8}{q^2+Q^2}\Big(\frac{2q(q-i(Q-1))}{(q-i(Q-1))^2-4u^2}-\frac{2q(q+i(Q-1))}{(q+i(Q-1))^2-4u^2}+\\ &-\frac{2q(q-i(Q+1))}{(q-i(Q+1))^2-4u^2}-\frac{2q(q+i(Q+1))}{(q+i(Q+1))^2-4u^2}-\frac{2(q^2+Q^2+2Q)}{1+4u^2}\Big)\end{aligned}$$

and one which contains polygamma functions

$$\frac{S_{0\psi}^{(2)}(q,Q,u)}{S_0^{(0)}(q,Q,u)} = \frac{16}{1+4u^2}\Big(\psi\big(\frac{1}{2}(-iq-Q)\big) - \psi\big(\frac{1}{2}(-iq+Q)\big)\Big) \tag{70}$$

Exponential part

The leading and subleading term of the exponent which appears in Y_Q is found to be

$$\left(\frac{z^-}{z^+}\right)^2 = \frac{16g^4}{(q^2+Q^2)^2}\left[1 - g^2\frac{16}{(q^2+Q^2)}\right]$$

where z^+ and z^- is taken from (68).

Dressing part

The dressing part can be found from formula (37) as

$$S_\sigma^{(2)}(q,Q,u) = -\frac{32}{1+4u^2}\Big(\gamma_E + \frac{1}{2}\psi\big(\frac{1}{2}(-iq-Q)\big) + \frac{1}{2}\psi\big(\frac{1}{2}(iq+Q)\big)\Big) \tag{71}$$

ABA modification

To make our notation more compact we rewrite the leading order Asymptotic Bethe Ansatz modification formula into the form

$$\Phi^{(8)} = \sum_{Q=1}^{\infty}\int_{-\infty}^{\infty}\frac{dq}{2\pi}\,\Phi_Q(q,u) = \sum_{Q=1}^{\infty}\int_{-\infty}^{\infty}\frac{dq}{2\pi}Y_Q^{(8)}(q,u)\mathrm{ABA}(q,Q,u) \tag{72}$$

In order to find $\mathrm{ABA}(q,Q,u)$ it is enough to use formulas (78)-(82) from [39]. Following (53) we have to take the derivative of S-matrix elements with respect to q and then calculate the supertrace. The final result we obtained is

$$\begin{aligned}\mathrm{ABA}(q,Q,u) = &-\frac{2q}{q^2+Q^2} + \frac{1}{-i-q-iQ+2u} + \frac{1}{i-q-iQ+2u} \\ &+\frac{1}{-i-q+iQ+2u} + \frac{1}{i-q+iQ+2u} - \frac{2(q+2u)}{1-q^2-Q^2+4u^2}\end{aligned}$$

The contribution $Y_Q^{(8,2)}(q)$

As mentioned in the previous section $Y_Q^{(8,2)}(q)$ comes from the 4-loop integrand expressed in terms of u, when we take into account the g^2 shifts of the rapidities of the constituents of the Konishi due to ABA. In order to find it we have to plug $u = \frac{1}{2\sqrt{3}} + \frac{4}{\sqrt{3}}g^2$ into $Y_Q^{(8)}(q)$ and expand it to the second order in g^2. The result is

$$Y_Q^{(8,2)}(q) = -\frac{4718592Q^2(16+9q^4-12Q^2+9Q^4+6q^2(-10+3Q^2))}{(9q^4+6q^2(2-6Q+3Q^2)+(4-6Q+3Q^2)^2)^2} \times \\ \times \frac{(-4+3q^2+3Q^2)(-16-9q^4-12Q^2+27Q^4+6q^2(-2+3Q^2))}{(9q^4+6q^2(2+6Q+3Q^2)+(4+6Q+3Q^2)^2)^2(q^2+Q^2)^4}$$

5.4 Integration

Before we proceed, it is fruitful to observe that we can symmetrize the integrand with respect to q without changing the result of integration

$$Y_Q^{sym}(q) = \frac{1}{2}(Y_Q(q) + Y_Q(-q)) \tag{73}$$

When we calculated the integrals we symmetrized some parts of the integrand leaving the rest not symmetrized depending on which form is easier to handle and gives simpler result.

Apart from the polygamma functions which appear in the dressing and scalar part, the remaining part of the integrand is a rational function. It can be then integrated over the real line by taking residues at the position of poles lying above the real line. All such poles can be classified into two groups: poles coming from the S-matrix parts of the integrand ('dynamical' poles), four of which lie above the real line:

$$q = i(Q \pm 1) \pm \frac{1}{\sqrt{3}} \;, \tag{74}$$

and a pole coming from the exponential part which is exactly $q = iQ$.

It turns out that when we symmetrize the whole integrand the contribution coming from dynamical poles vanishes when summed over Q as in the 4-loop case[10]. It can be explained by the fact that at weak coupling we

[10]It can be checked both numerically and completely algebraically using the methods of the subsequent section.

expect the μ-term contributions to be absent. We want to stress that this is a far from trivial consistency check of our formulas.

Beside the residues of the rational function poles we have to handle the additional poles coming from the polygamma functions which appear in the dressing and scalar part. In order to simplify our considerations, it is convenient to symmetrize the $S^{(2)}_{0\psi}(q,Q,u)$ part. It turns out that in that case the polygamma functions will vanish leaving us only with rational function

$$\Big(\psi\big(\frac{1}{2}(-iq-Q)\big)-\psi\big(\frac{1}{2}(-iq+Q)\big)\Big)^{sym}=-\frac{2Q}{q^2+Q^2}$$

So the only remaining polygamma functions come from the dressing part. The positions of polygamma function poles are known to be at negative integers. In our case it means that $\frac{1}{2}(-iq-Q)=-n$ or $\frac{1}{2}(iq+Q)=-n$ where $n\geq 0$. Solving with respect to q we obtain the positions of integrand poles coming from polygamma function as

$$q=i(Q-2n)\quad n\geq 0 \qquad \text{or} \qquad q=i(Q+2n)\quad n\geq 0 \tag{75}$$

We want to close the contour of integration over the real line meaning we have to take into account only those residues which have nonnegative imaginary part.

In the proceeding sections we will calculate all the residues. We have chosen to symmetrize only two parts of integrand: the scalar integrand part which contains polygamma functions and the part coming from the ABA modification. The rest is left in a nonsymmetrized form.

Residues of rational functions

For the rational part of the integrand the result after taking the residues (regardless if we take $q=iQ$ or 'dynamical' poles) can be rewritten as a sum of terms with minimal denominators which are of the form

$$\frac{a}{Q^n} \tag{76}$$

which will give zeta functions when summed up or

$$\frac{aQ+b}{1\pm 3Q+3Q^2} \tag{77}$$

which will produce polygamma functions.

For the dressing part the residue at $q = iQ$ gives exactly

$$\frac{3456(-2+12Q^2-45Q^4+27Q^6)\zeta(3)}{Q^3(1-3Q^2+9Q^4)^2} \tag{78}$$

while the residues at the dynamical poles are some rational functions of Q with complicated coefficients containing $\psi(\frac{1}{6}(3+i\sqrt{3}))$ and $\psi(\frac{1}{6}(3-i\sqrt{3}))$.

Residues of polygamma functions

To calculate the contribution coming from the polygamma functions appearing in the dressing part of the integrand we want to find all the poles of the form (75) which have nonnegative imaginary part. It is easy to notice that such poles are of the form $q_{Q,n} = i(Q+2n)$ where n is (possibly negative) integer obeying $Q + 2n \geq 0$. The residues at the positions $q_{Q,n}$ is found to be

$$\frac{-864Q^2(1+3n^2+3nQ)^2\mathrm{sign}(n)}{n^4(1-3n^2+9n^4)(n+Q)^4\big((1+3n^2+6nQ+3Q^2)^2-(3n+3Q)^2\big)}$$

The remaining task is to sum the above formula over n and Q. We have to be careful during the summation process because there exist one pole which lies on the real line for every even Q. In that case we have to take only half of the residue with the minus sign due to how we have chosen the orientation of our integration contour. Additionally, we have to remember that the residues at $q = iQ$ were taken into account before. Keeping it in mind we calculated the sum over Q and obtained

$$\frac{1728(3n^2-1)(-1+3n^2+3n^3-9n^4+9n^5+(n^3-3n^5+9n^7)\psi^{(2)}(1+n))}{n^6(1-3n^2+9n^4)^2}$$

which have to be summed over n from 1 to ∞. It can be done using methods from the next section.

6 Summation over bound-states

There are two types of sums over bound-states appearing in our calculations after the residues are taken. Firstly, there are sums of the form (76) or (77)

which can be easily summed using *Mathematica*. The first ones give us ζ-functions while the second ones contain polygamma functions of the form $\psi(\frac{1}{6}(3+i\sqrt{3}))$ and $\psi(\frac{1}{6}(3-i\sqrt{3}))$.

On the other hand during the 5-loop Konishi calculation more difficult sums emerge

$$\Sigma^{(m)} = \sum_{Q=1}^{\infty} R(Q)\,\psi^{(m)}(Q) \qquad m \geq 0, \tag{79}$$

where $R(x)$ is a rational function of x and $\psi^{(m)}(x)$ is the mth polygamma function given by the definition

$$\psi^{(m)}(x) = \frac{d^m\,\psi(x)}{d\,x^m}, \qquad m \geq 1, \tag{80}$$

with $\psi(x) \equiv \psi^{(0)}(x)$ being the digamma function $\psi(x) = \frac{d\,\log\Gamma(x)}{d\,x}$. The evaluation of the sum (79) goes as follows: $R(Q)$ is decomposed as a sum of two terms $R(Q) = R_0(Q) + R_1(Q)$, where $R_0(Q)$ contains the sum of pure power terms of the partial fraction decomposition of $R(Q)$ (i.e. $R_0(Q) = \frac{a_1}{Q^{n_1}} + \frac{a_2}{Q^{n_2}} + ...$), while $R_1(Q)$ contains the rest. In this case the sum (79) is decomposed into two parts as well:

$$\Sigma^{(m)} = \Sigma_0^{(m)} + \Sigma_1^{(m)}, \tag{81}$$

where

$$\Sigma_a^{(m)} = \sum_{Q=1}^{\infty} R_a(Q)\,\psi^{(m)}(Q) \qquad m \geq 0, \qquad a = 0,1. \tag{82}$$

Using a series representation for the polygamma functions, $\Sigma_0^{(m)}$ can be expressed in terms of the values at infinity of nested harmonic sums, which can be expressed in terms of multivariate zeta functions (Zagier-Euler sums) [43]. These sums can be reexpressed in terms of ordinary Euler sums. The relations between the former and the latter can be found using EZ-Face - an online calculator for Euler sums [44]. At the end of the process $\Sigma_0^{(m)}$ is expressed in terms of zeta functions taken at integer values.

The calculation of $\Sigma_1^{(m)}$ requires a different method: representing $R_1(x)$ as the Laplace transform of its inverse Laplace transform, and using appropriate integral representation for the polygamma functions, $\Sigma_1^{(m)}$ can be transformed into a double integral form, in which the summation can be easily performed

and the remaining double integral expression can be calculated exactly with the help of *Mathematica*. For details see Appendix B.

Using the methods presented above all the sums can be evaluated. The striking observation is that all 'nasty' polygamma functions disappear leaving us only with the ζ-functions. The final result is

$$\Delta_w^{(10)} = -11340 + 2592\zeta(3) - 5184\zeta(3)^2 - 11520\zeta(5) + 30240\zeta(7) \qquad (83)$$

As for the 4-loop case the result is a sum of ζ-functions of odd degrees with integer coefficients and correct transcendentality degree. The new feature of the result is that products of ζ-functions start to appear which have been absent in the 4-loop case.

To summarize the anomalous dimension of the Konishi operator up to 5-loops is

$$\begin{aligned} \Delta &= 4 + 12g^2 - 48g^4 + 336g^6 + 96(-26 + 6\zeta(3) - 15\zeta(5))g^8 \\ &\quad -96(-158 - 72\zeta(3) + 54\zeta(3)^2 + 90\zeta(5) - 315\zeta(7))g^{10} \end{aligned} \qquad (84)$$

7 Conclusions

Anomalous dimensions of operators in $\mathcal{N} = 4$ SYM correspond to energies of string states in $AdS_5 \times S^5$. The leading perturbative orders are given by the Asymptotic Bethe Ansatz, while at a certain loop order, there appear new contributions coming from a topologically distinct class of Feynman diagrams – so-called 'wrapping interactions'. On the string theory side these correspond to virtual particles propagating around the string worldsheet cylinder. The leading corrections arise from multiparticle generalizations of the classical Lüscher terms. Further corrections are due to many virtual particles and all these should in principle be resummed by a TBA system.

The most convenient testing ground for these issues is the shortest non-protected operator in $\mathcal{N} = 4$ SYM – the Konishi operator, as well as single impurity operators in the β-deformed theory.

In a previous paper [39], the leading four loop part of the wrapping correction has been found. It came from a two-particle Lüscher term and gave

$$\Delta^{(8)}_{w,Konishi} = 324 + 864\,\zeta(3) - 1440\,\zeta(5) \qquad (85)$$

The aim of the present paper was to compute the five loop wrapping contribution. Although it still arises just from the two-particle Lüscher term, it is very interesting as it involves two new ingredients. Firstly, there is a nontrivial modification of the Asymptotic Bethe Ansatz quantization due to the sea of virtual particles. This term is quite interesting as it does not seem to be simply related to the transfer matrix appearing in a direct expansion of the proposed TBA systems. Secondly, the BES/BHL dressing factor cannot be neglected any more, and moreover, due to the specific kinematics of the Lüscher term, an *infinite* set of coefficients of the dressing phase starts to contribute at once. The result obtained in the present paper is

$$\Delta^{(10)}_{w,Konishi} = -11340 + 2592\,\zeta(3) - 5184\,\zeta(3)^2 - 11520\,\zeta(5) + 30240\,\zeta(7) \quad (86)$$

This computation is subject to two nontrivial cross-checks. Firstly, the partial results (coming from the dressing phase, modification of the ABA and the remaining part of the S-matrix) have a very complicated transcendentality structure involving polygamma functions. All these cancel out leaving the final result as a simple combination of odd ζ functions. Secondly, the residues of the 'dynamical' poles (associated with μ-terms) should cancel out between the various terms when summed over the types of bound-states. Both of these cross-checks are satisfied in our case and involve a rather intricate conspiracy between the various terms.

In Appendix A, we have computed the five loop wrapping correction to a single impurity in the β-deformed theory[11]. In this case the four loop result was ($M = 1$ in [32], see also [45, 46])

$$\Delta^{(8)}_{w,single} = 496\,\zeta(3) - 640\,\zeta(5) \quad (87)$$

while the five loop wrapping contribution computed in Appendix A is

$$\Delta^{(10)}_{w,single} = -1536\,\zeta(3)^2 - 4096\,\zeta(3) - 5120\,\zeta(5) + 13440\,\zeta(7) \quad (88)$$

The ABA modification does not appear here, however the *infinite* set of BES coefficients contribute just as in the case of the Konishi operator.

It would be very interesting to verify these results perturbatively. Especially the single impurity operator may be within reach. This might help

[11] This computation relies on an additional assumption on performing analytical continuation from even to odd spins in [32]. The four loop result has been verified by a direct perturbative computation in [41].

in understanding the precise relation between the structure of direct perturbative computations and the string theory computations based on the S-matrices and Lüscher formulas.

Another application of these results would be to test the excited state TBA systems recently proposed for AdS/CFT. It would be very interesting to see the appearance of the rather intricate modification of the Asymptotic Bethe Ansatz and the specific analytical continuation of the dressing phase from these formulations, especially as both of these ingredients are very strongly constrained by the transcendentality structure and the cancellation of dynamical poles between themselves and the rest of the integrand. The four loop result is sensitive to the TBA source terms and has been rederived in [35]. The five loop result depends on the structure of the convolution terms and thus is a more sensitive test of excited state TBA systems. In particular the procedure of section 3 to obtain the Lüscher corrections could be applied e.g. to the TBA system in the second paper of [35]. However this is technically quite involved. The five loop wrapping correction for the Konishi operator seems therefore to be an interesting and robust test for the excited TBA systems.

Acknowledgments: We would like to thank Sergey Frolov, Christoph Sieg and Dima Volin for interesting discussions related to this project. R.J. and T.L. were supported by Polish science funds during 2009-2011 as a research project (NN202 105136). The work was supported by Marie Curie Transfer of Knowledge Project COCOS (MTKD-CT-2004-517186). R.J. was supported by the RTN network ENRAGE MRTN-CT-2004-005616. Z.B. was supported by a Bolyai Research Fellowship and by the Hungarian National Science Fund OTKA T60040 and A.H. by OTKA T049495.

A Lüscher correction for a one particle state

In this Appendix we calculate the subleading wrapping correction to the energy of a one particle state. Such a state must have vanishing momentum in the $\mathcal{N}=4$ super-Yang-Mills theory thus its energy is protected. In the β deformed theory, however, at $\beta=\frac{1}{2}$ a one particle state with vanishing rapidity $u=0$ ($p=\pi$) is allowed and acquires nontrivial finite size corrections. The condition for its vanishing rapidity is protected at all orders both in the ABA and in the wrapping corrections. So in this case we will not have a contribution coming from the modification of the ABA quantization. However there will be a nontrivial direct wrapping correction to the energy, which will include, starting from five loops, a contribution from the BES dressing phase. Thus the single impurity operator provides a testing ground for that part of the Lüscher correction.

The leading order wrapping correction can be written as

$$\Delta E_w = -\sum_{Q=1}^{\infty}\int\frac{dq}{2\pi}(-1)^F S_{Q1}^{Q1}(q,0)e^{-\tilde{\epsilon}(q)L} = -\sum_{Q=1}^{\infty}\int\frac{dq}{2\pi}Y_Q(q) \tag{89}$$

Here S_{Q1}^{Q1} represents the scattering matrix of the mirror Q particle with the fundamental $u=0$ physical particle. We can further decompose the scattering part as the scalar part (S_0), the dressing part (S_σ) and the matrix part ($S_\boxplus$):

$$S_{Q1}^{Q1}(q,0) = S_0(q,Q)S_\sigma(q,Q)S_\boxplus(q,Q)^2 \tag{90}$$

We expand each quantity to subleading order in g^2 as

$$\begin{aligned}
S_0(q,Q) &= S_0^{(0)}(q,Q)+g^2S_0^{(2)}(q,Q)+\dots\\
S_\sigma(q,Q) &= 1+g^2S_\sigma^{(2)}(q,Q)+\dots\\
S_\boxplus(q,Q) &= g^2S_\boxplus^{(2)}(q,Q)+g^4S_\boxplus^{(4)}(q,Q)+\dots
\end{aligned}$$

When we calculated the leading wrapping correction to twist two operators for odd particle number [32], we observed that in order to be compatible with gauge theory calculations and to provide the proper analytical continuation from even cases to odd ones we had to omit the contribution of the fermions. We accept this convention now too, but call the attention for the need of a derivation of this proposal based on first principles. Under this assumption the leading order correction of the matrix part is

$$S_\boxplus^{(2)}(q,Q) = -\frac{16Q(q^2+Q^2-1)}{(q^2+Q^2)(Q+iq-1)} \tag{91}$$

while the subleading one is

$$\frac{S_{\boxplus}^{(4)}(q,Q)}{S_{\boxplus}^{(2)}(q,Q)} = -\frac{4(q^3 - iq^2(Q-3) - i(Q-1)^3 + q(Q^2+1))}{(q^2+Q^2)(q-i(Q-1))} \tag{92}$$

The leading and subleading correction of the exponential part is given by

$$e^{-\tilde{\epsilon}(q)} = g^4\Upsilon^{(4)}(q,Q) + g^6\Upsilon^{(6)}(q,Q) = \frac{16g^4}{(q^2+Q^2)^2}\left[1 - g^2\frac{16}{(q^2+Q^2)}\right] \tag{93}$$

In calculating the scalar part we have to use the parameterization of the bound-state as we did for the Konishi operator. The result for the leading order scalar part reads as

$$S_0^{(0)}(q,Q) = -\frac{q-i(Q-1)}{(q+i(Q-1))(q-i(Q+1))(q+i(Q+1))} \tag{94}$$

while the subleading contains a rational part

$$\begin{aligned} \frac{S_{0rat}^{(2)}(q,Q)}{S_0^{(0)}(q,Q)} &= 8 + \frac{16(-1+2iq)q^2 - 16(2+q(i+6q))Q}{(1+4q^2)(q^2+Q^2)} \\ &\quad + \frac{16qi(-1+2q^2+Q)}{(1+4q^2)(q^2+(-1+Q)^2)} - \frac{2q(3+2Q)}{(1+4q^2)(q^2+(1+Q)^2))} \end{aligned}$$

and a polygamma part:

$$\frac{S_{0\psi}^{(2)}(q,Q)}{S_0^{(0)}(q,Q)} = 8\psi(-\frac{1}{2}(iq+Q)) - 8\psi(\frac{1}{2}(-iq+Q)) \tag{95}$$

The dressing part reads as

$$S_\sigma(q,Q) = -8\left[2\gamma_E + \psi(-\frac{1}{2}(iq+Q)) + \psi(\frac{1}{2}(iq+Q))\right] \tag{96}$$

The full integrand can be written as

$$Y_Q(g) = g^8 Y_Q^{(8)}(q) + g^{10} Y_Q^{(10)}(q) + \dots \tag{97}$$

The leading order part is given by

$$\begin{aligned} Y_Q^{(8)}(q) &= S_0^{(0)}(q,Q) S_{\boxplus}^{(2)}(q,Q)^2 \Upsilon^{(4)}(q,Q) \\ &= -\frac{4096Q^2(-1+q^2+Q^2)^2}{(q^2+Q^2)^4(q^4+(-1+Q^2)^2+2q^2(1+Q^2))} \end{aligned}$$

It has a kinematical pole at iQ and two dynamical poles at $i(Q \pm 1)$ on the upper half plane. If we take the residue of the integrand only at the kinematical pole and sum over Q we obtain the leading wrapping correction

$$\Delta_w^{(8)} = 128(4\zeta(3) - 5\zeta(5)) \tag{98}$$

We note that the contributions of the dynamical poles summed over Q cancel out, so this is indeed the full leading order result, which has been verified by a direct perturbative calculation on the gauge theory side in [41].

In calculating the 5-loop subleading correction we can write

$$Y_Q^{(10)}(q) = Y_Q^{(8)}(q) \left[\frac{S_0^{(2)}(q,Q)}{S_0^{(0)}(q,Q)} + 2\frac{S_\boxplus^{(4)}(q,Q)}{S_\boxplus^{(2)}(q,Q)} + S_\sigma^{(2)}(q,Q) + \frac{\Upsilon^{(6)}(q,Q)}{\Upsilon^{(4)}(q,Q)} \right]$$

We analyze separately the contributions of the rational part and the polygamma part. We can observe that the rational part of the scalar contribution together with the matrix contribution and the exponential part gives a symmetric function in q:

$$\begin{aligned} \frac{S_{0rat}^{(2)}(q,Q)}{S_0^{(0)}(q,Q)} + 2\frac{S_\boxplus^{(4)}(q,Q)}{S_\boxplus^{(2)}(q,Q)} + \frac{\Upsilon^{(6)}(q,Q)}{\Upsilon^{(4)}(q,Q)} \quad &= \\ -\frac{(8(q^4(7+2Q) + (3+2Q)(-1+Q^2)^2 + 2q^2(5+2Q)(1+Q^2))}{(q^2+(-1+Q)^2)(q^2+Q^2)(q^2+(1+Q)^2))} & \end{aligned}$$

Taking the residue at the kinematical pole and summing over Q gives

$$\frac{128}{27}(72\pi^2 - 30\pi^4 + 2\pi^6 - 1296\zeta(3) - 1080\zeta(5) + 2835\zeta(7)) \tag{99}$$

The contributions of the dynamical poles do not cancel when summed over Q (as we did not analyze yet the full expression) and give

$$-\frac{256}{3}(-12\pi^2 + \pi^4) \tag{100}$$

The polygamma part is a bit more complicated as it has poles not only at the kinematical and dynamical locations but additionally at[12] $q = i(Q + 2n)$,

[12] In contrast to the Konishi case we have not symmetrized the polygamma part of scalar integral. This choice leads to a slightly different configuration of the poles.

$n > 0$. The contribution of its residues are

$$\underset{q=i(Q+2n)}{\mathrm{Res}} Y_Q^{(8)}(q) \left[\frac{S_{0\psi}^{(2)}(q,Q)}{S_0^{(0)}(q,Q)} + S_\sigma^{(2)}(q,Q) \right] = -\sum_{n=1}^{\infty} \frac{256 Q^2 (1+4n(n+Q))^2}{n^4(-1+4n^2)(n+Q)^4(-1+4(n+Q)^2))}$$

This can be summed over n resulting in polygamma functions. We combine this result with the contribution of the remaining poles obtained by taking residues of $Y_Q^{(8)}(q) \left[\frac{S_{0\psi}^{(2)}(q,Q)}{S_0^{(0)}(q,Q)} + S_\sigma^{(2)}(q,Q) \right]$ at the kinematical and dynamical poles. The final expression can be summed over Q by the methods of Appendix B. The result is simply

$$\frac{256}{27}(-\pi^2(-12+\pi^2)^2 + 54(4-3\zeta(3))\zeta(3)) \tag{101}$$

When we combine this result with the result of the rational part we can observe that the even ζ part cancels out and we arrive at the subleading wrapping correction

$$\Delta_w^{(10)} = -128(12\zeta(3)^2 + 32\zeta(3) + 40\zeta(5) - 105\zeta(7)) \tag{102}$$

We checked that the dynamical residue contributions of the full integrand cancel when summed over Q as in the Konishi case providing another convincing support for our considerations.

B Summation of terms containing polygamma functions

Here we present a method which enables one to calculate sums containing polygamma functions. A typical sum emerging during the 5-loop Konishi computation is of the form:

$$\Sigma^{(m)} = \sum_{Q=1}^{\infty} R(Q)\, \psi^{(m)}(Q) \qquad m \geq 0, \tag{103}$$

where $R(x)$ is a rational function of x and $\psi^{(m)}(x)$ is the mth polygamma function given by the definition

$$\psi^{(m)}(x) = \frac{d^m\,\psi(x)}{d\,x^m}, \qquad m \geq 1, \tag{104}$$

with $\psi(x) \equiv \psi^{(0)}(x)$ being the digamma function $\psi(x) = \frac{d\,\log\Gamma(x)}{d\,x}$. The evaluation of sum (103) goes as follows: $R(Q)$ is decomposed as a sum of two terms $R(Q) = R_0(Q) + R_1(Q)$, where $R_0(Q)$ contains the sum of pure inverse power terms of the partial fraction decomposition of $R(Q)$ (i.e. $R_0(Q) = \frac{a_1}{Q^{n_1}} + \frac{a_2}{Q^{n_2}} + ...$), while $R_1(Q)$ contains the rest. In this case the sum (103) is decomposed into 2 parts as well:

$$\Sigma^{(m)} = \Sigma_0^{(m)} + \Sigma_1^{(m)}, \tag{105}$$

where

$$\Sigma_a^{(m)} = \sum_{Q=1}^{\infty} R_a(Q)\,\psi^{(m)}(Q) \qquad m \geq 0, \qquad a = 0, 1. \tag{106}$$

Using a series representation for the polygamma functions the sum $\Sigma_0^{(m)}$ can be evaluated directly applying the method sketched in section 6. So, hereafter we concentrate on the calculation of the sums $\Sigma_1^{(m)}$. They are evaluated by transforming them into integral expressions calculable with the help of *Mathematica*. The transformation is as follows: the polygamma functions are represented by their appropriate integral representations

$$\psi^{(m)}(x) = (-1)^{m+1} \int_0^{\infty} dt\, \frac{t^m\, e^{-xt}}{1 - e^{-t}}, \qquad m \geq 1, \tag{107}$$

$$\psi(x) \equiv \psi^{(0)}(x) = \int_0^{\infty} dt\, \left(\frac{e^{-t}}{t} - \frac{e^{-xt}}{1 - e^{-t}} \right). \tag{108}$$

and the function $R_1(x)$ is represented as the Laplace transform of its inverse Laplace transform

$$R_1(x) = \int_0^{\infty} dt\, e^{-xt}\, \mathcal{L}R_1^{-1}(t), \tag{109}$$

where $\mathcal{L}R_1^{-1}(t)$ stands for the inverse Laplace transform of $R_1(x)$ given by the formula

$$\mathcal{L}R_1^{-1}(t) = \frac{1}{2\pi i} \int\limits_{\eta-i\infty}^{\eta+i\infty} ds\, e^{st}\, R_1(s), \tag{110}$$

with η being an arbitrary positive constant chosen so that the contour of integration lies to the right of all singularities in $R_1(s)$. Due to the structural difference between integral representations (107) and (108) one has to make a distinction between the cases $m \geq 1$ and $m = 0$ and consider them separately.

<u>$m \geq 1$ **case:**</u>

Using integral representations (107) and (109), $\Sigma_1^{(m)}$ takes the form

$$\Sigma_1^{(m)} = \sum_{Q=1}^{\infty} \int\limits_0^{\infty} dt\, e^{-Qt}\, \mathcal{L}R_1^{-1}(t)\, (-1)^{m+1} \int\limits_0^{\infty} dt' \, \frac{t'^m\, e^{-Qt'}}{1-e^{-t'}}$$

which after evaluating the simple geometric sum in Q becomes

$$\int\limits_0^{\infty} dt\, \mathcal{L}R_1^{-1}(t)\, (-1)^{m+1} \int\limits_0^{\infty} dt' \, \frac{t'^m}{(1-e^{-t'})(e^{t+t'}-1)}$$

Then using the identity

$$\int\limits_0^{\infty} dt' \, \frac{t'^m}{(1-e^{-t'})(e^{t+t'}-1)} = -\Gamma(m+1)\, \frac{\mathrm{Li}_{m+1}(e^{-t}) - \zeta(1+m)}{e^t - 1}, \tag{111}$$

where $\mathrm{Li}_n(x)$ is the nth polylogarithm function, the integral with respect to t' can be evaluated and finally a single integral remains

$$\Sigma_1^{(m)} = (-1)^m \int\limits_0^{\infty} dt\, \mathcal{L}R_1^{-1}(t)\, \Gamma(m+1)\, \frac{\mathrm{Li}_{m+1}(e^{-t}) - \zeta(1+m)}{e^t - 1}. \tag{112}$$

During the Konishi computation m took the values of $1, 2$ and 3 and in all cases emerging during the calculations, the integrals (112) could be evaluated by *Mathematica*.

<u>$m = 0$ **case**</u>

Representing $R_1(x)$ as the Laplace transform of its inverse Laplace transform, and using integral representation (108) for $\psi^{(0)}(x)$ the sum takes the form:

$$\Sigma_1^{(0)} = \sum_{Q=1}^{\infty} \int_0^{\infty} dt\, e^{-Qt}\, \mathcal{L}R_1^{-1}(t) \int_0^{\infty} dt' \left(\frac{e^{-t'}}{t'} - \frac{e^{-Qt'}}{1 - e^{-t'}} \right)$$

Again evaluating the simple geometric sum in Q, this can be recast as

$$\int_0^{\infty} dt\, \mathcal{L}R_1^{-1}(t) \int_0^{\infty} dt' \left(\frac{e^{-t'}}{t' \,(e^{t} - 1)} - \frac{1}{(1 - e^{-t'})\,(e^{t+t'} - 1)} \right)$$

Now exploiting the integral formula

$$\int_0^{\infty} dt' \left(\frac{e^{-t'}}{t' \,(e^{t} - 1)} - \frac{1}{(1 - e^{-t'})\,(e^{t+t'} - 1)} \right) = \frac{\gamma_E + \log(1 - e^{-t})}{1 - e^{t}}, \qquad (113)$$

with γ_E being the Euler's constant, the integral with respect to t' can be evaluated and one ends up with an expression containing a single integral

$$\Sigma_1^{(0)} = \int_0^{\infty} dt\, \mathcal{L}R_1^{-1}(t)\, \frac{\gamma_E + \log(1 - e^{-t})}{1 - e^{t}}. \qquad (114)$$

During Konishi computations this formula made it possible to evaluate sums containing $\psi^{(0)}(x)$ functions.

Finally, we just note that in the case of $m = 0$ the sum $\Sigma_0^{(0)}$ can also be evaluated by the application of formula (114).

References

[1] J. M. Maldacena, "The large N limit of superconformal field theories and supergravity," Adv. Theor. Math. Phys. **2** (1998) 231 [Int. J. Theor. Phys. **38** (1999) 1113], [hep-th/9711200];
S. S. Gubser, I. R. Klebanov and A. M. Polyakov, "Gauge theory correlators from non-critical string theory," Phys. Lett. B **428** (1998) 105, [hep-th/9802109];
E. Witten, "Anti-de Sitter space and holography," Adv. Theor. Math. Phys. **2** (1998) 253, [hep-th/9802150].

[2] I. Bena, J. Polchinski and R. Roiban, "Hidden symmetries of the AdS(5) x S**5 superstring," Phys. Rev. D **69** (2004) 046002, [hep-th/0305116].

[3] C. Itzykson and J. B. Zuber, "Quantum Field Theory," *New York, Usa: Mcgraw-hill (1980) 705 P.(International Series In Pure and Applied Physics)*

[4] A. B. Zamolodchikov and A. B. Zamolodchikov, "Factorized S-matrices in two dimensions as the exact solutions of certain relativistic quantum field models," Annals Phys. **120**, 253 (1979).

[5] P. Dorey, "Exact S matrices," arXiv:hep-th/9810026.

[6] S. Frolov and R. Suzuki, arXiv:0906.0499 [hep-th].

[7] M. Luscher, "Volume Dependence of the Energy Spectrum in Massive Quantum Field Theories. 2. Scattering States," Commun. Math. Phys. **105**, 153 (1986).

[8] M. Luscher, "Volume Dependence Of The Energy Spectrum In Massive Quantum Field Theories. 1. Stable Particle States," Commun. Math. Phys. **104**, 177 (1986).

[9] A. B. Zamolodchikov, "Thermodynamic bethe ansatz in relativistic models. Scaling three state Potts and Lee-Yang models," Nucl. Phys. B **342**, 695 (1990).

[10] P. Dorey and R. Tateo, "Excited states by analytic continuation of TBA equations," Nucl. Phys. B **482**, 639 (1996) [arXiv:hep-th/9607167].

[11] J. A. Minahan and K. Zarembo, "The Bethe-ansatz for N = 4 super Yang-Mills," JHEP **0303**, 013 (2003), [hep-th/0212208].

[12] N. Beisert, C. Kristjansen and M. Staudacher, "The dilatation operator of N = 4 super Yang-Mills theory," Nucl. Phys. B **664** (2003) 131, [hep-th/0303060].

[13] N. Beisert and M. Staudacher, "The N = 4 SYM integrable super spin chain," Nucl. Phys. B **670** (2003) 439, [hep-th/0307042].

[14] N. Beisert, V. Dippel and M. Staudacher, "A novel long range spin chain and planar N = 4 super Yang-Mills," JHEP **0407**, 075 (2004) [hep-th/0405001].

[15] N. Beisert, V. A. Kazakov, K. Sakai and K. Zarembo, "Complete spectrum of long operators in N = 4 SYM at one loop," JHEP **0507**, 030 (2005) [arXiv:hep-th/0503200].

[16] V. A. Kazakov, A. Marshakov, J. A. Minahan and K. Zarembo, "Classical / quantum integrability in AdS/CFT," JHEP **0405**, 024 (2004) [arXiv:hep-th/0402207].

[17] G. Arutyunov, S. Frolov and M. Staudacher, "Bethe ansatz for quantum strings," JHEP **0410**, 016 (2004) [arXiv:hep-th/0406256].

[18] M. Staudacher, "The factorized S-matrix of CFT/AdS," JHEP **0505**, 054 (2005) [hep-th/0412188].

[19] N. Beisert and M. Staudacher, "Long-range PSU(2,2|4) Bethe ansaetze for gauge theory and strings," Nucl. Phys. B **727**, 1 (2005) [hep-th/0504190].

[20] N. Beisert, "The su(2—2) dynamic S-matrix," Adv. Theor. Math. Phys. **12**, 945 (2008) [arXiv:hep-th/0511082].

[21] A. Rej, D. Serban and M. Staudacher, "Planar N = 4 gauge theory and the Hubbard model," JHEP **0603**, 018 (2006) [arXiv:hep-th/0512077].

[22] R. A. Janik, "The AdS(5) x S**5 superstring worldsheet S-matrix and crossing symmetry," Phys. Rev. D **73**, 086006 (2006) [arXiv:hep-th/0603038].

[23] G. Arutyunov, S. Frolov and M. Zamaklar, "The Zamolodchikov-Faddeev algebra for AdS(5) x S**5 superstring," JHEP **0704**, 002 (2007) [arXiv:hep-th/0612229].

[24] N. Beisert, R. Hernandez and E. Lopez, "A crossing-symmetric phase for AdS(5) x S**5 strings," JHEP **0611**, 070 (2006) [arXiv:hep-th/0609044].

[25] N. Beisert, B. Eden and M. Staudacher, "Transcendentality and crossing," J. Stat. Mech. **0701**, P021 (2007) [arXiv:hep-th/0610251].

[26] D. Volin, "Minimal solution of the AdS/CFT crossing equation," arXiv:0904.4929 [hep-th].

[27] N. Dorey and K. Okamura, "Singularities of the Magnon Boundstate S-Matrix," JHEP **0803**, 037 (2008) [arXiv:0712.4068 [hep-th]]. H. Y. Chen, N. Dorey and K. Okamura, "On the scattering of magnon boundstates," JHEP **0611** (2006) 035 [arXiv:hep-th/0608047].

[28] G. Arutyunov and S. Frolov, "The S-matrix of String Bound States," Nucl. Phys. B **804** (2008) 90 [arXiv:0803.4323 [hep-th]].

[29] N. Dorey, D. M. Hofman and J. M. Maldacena, "On the singularities of the magnon S-matrix," Phys. Rev. D **76**, 025011 (2007) [arXiv:hep-th/0703104].

[30] J. Ambjorn, R. A. Janik and C. Kristjansen, "Wrapping interactions and a new source of corrections to the spin-chain / string duality," Nucl. Phys. B **736**, 288 (2006) [arXiv:hep-th/0510171].

[31] R. A. Janik and T. Lukowski, "Wrapping interactions at strong coupling – the giant magnon," Phys. Rev. D **76**, 126008 (2007) [arXiv:0708.2208 [hep-th]].

[32] Z. Bajnok, R. A. Janik and T. Lukowski, "Four loop twist two, BFKL, wrapping and strings," Nucl. Phys. B **816** (2009) 376 [arXiv:0811.4448 [hep-th]].

[33] G. Arutyunov and S. Frolov, "On String S-matrix, Bound States and TBA," JHEP **0712**, 024 (2007) [arXiv:0710.1568 [hep-th]].

[34] D. Bombardelli, D. Fioravanti and R. Tateo, "Thermodynamic Bethe Ansatz for planar AdS/CFT: a proposal," arXiv:0902.3930 [hep-th].

[35] N. Gromov, V. Kazakov and P. Vieira, "Integrability for the Full Spectrum of Planar AdS/CFT," arXiv:0901.3753 [hep-th];
N. Gromov, V. Kazakov, A. Kozak and P. Vieira, "Integrability for the Full Spectrum of Planar AdS/CFT II," arXiv:0902.4458 [hep-th].

[36] G. Arutyunov and S. Frolov, "String hypothesis for the AdS_5xS^5 mirror," JHEP **0903**, 152 (2009) [arXiv:0901.1417 [hep-th]];
G. Arutyunov and S. Frolov, "Thermodynamic Bethe Ansatz for the AdS_5xS^5 Mirror Model," JHEP **0905** (2009) 068 [arXiv:0903.0141 [hep-th]],

[37] F. Fiamberti, A. Santambrogio, C. Sieg and D. Zanon, "Wrapping at four loops in N=4 SYM," arXiv:0712.3522 [hep-th].

[38] V. N. Velizhanin, "The Four-Loop Konishi in N=4 SYM," [arXiv:hep-th/0808.3832]

[39] Z. Bajnok and R. A. Janik, "Four-loop perturbative Konishi from strings and finite size effects for multiparticle states," Nucl. Phys. B **807** (2009) 625 [arXiv:0807.0399 [hep-th]].

[40] A. V. Kotikov, L. N. Lipatov, A. Rej, M. Staudacher and V. N. Velizhanin, "Dressing and Wrapping," J. Stat. Mech. **0710** (2007) P10003 [arXiv:0704.3586 [hep-th]].

[41] F. Fiamberti, A. Santambrogio, C. Sieg and D. Zanon, "Finite-size effects in the superconformal beta-deformed N=4 SYM," JHEP **0808** (2008) 057 [arXiv:0806.2103 [hep-th]];
F. Fiamberti, A. Santambrogio, C. Sieg and D. Zanon, "Single impurity operators at critical wrapping order in the beta-deformed N=4 SYM," arXiv:0811.4594 [hep-th].

[42] G. Arutyunov and S. Frolov, "The Dressing Factor and Crossing Equations," arXiv:0904.4575 [hep-th].

[43] A.V. Kotikov, V.N. Velizhanin, Analytic continuation of the Mellin moments of deep inelastic structure functions, arXiv:hep-ph/0501274

[44] See `http://oldweb.cecm.sfu.ca/projects/EZFace/`

[45] J. Gunnesson, "Wrapping in maximally supersymmetric and marginally deformed N=4 Yang-Mills," arXiv:0902.1427 [hep-th].

[46] M. Beccaria and G. F. De Angelis, "On the wrapping correction to single magnon energy in twisted N=4 SYM," arXiv:0903.0778 [hep-th].